Exercise Set #6

"Discrete Mathematics" (2025)

Exercise 7 is to be submitted on Moodle before 23:59 on March 31st, 2025

E1. Prove the following identities for Fibonacci numbers. In each identity below $n \geq 1$.

- **(a)** $F_1 + F_3 + F_5 \ldots + F_{2n-1} = F_{2n}$.
- **(b)** $F_{2n+1} = 3F_{2n-1} - F_{2n-3}$.
- **(c)** $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$.
- **(d)** $\gcd(F_n, F_{n+1}) = 1$.

E2. Prove that any positive integer can be written as a sum of mutually distinct Fibonacci numbers.

E3. Consider the sequence $(a_0, a_1, a_2 \ldots)$ with $a_0 = 1, a_1 = 2, a_2 = 3$ and

$$a_{k+1} = 5a_k - 8a_{k-1} + 4a_{k-2}$$

for $k \geq 2$. Find an expression for the value of a_k. What is its generating function?

E4. What is the generating function of the sequence $(a_0, a_1, a_2 \ldots)$ with $a_0 = 1, a_1 = 3$ and $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geq 2$?

E5. Suppose that $a_0 = 2, a_1 = 8$ and for $n \geq 0$ we have $a_{n+2} = \sqrt{a_n a_{n+1}}$. Can you write an expression for the general a_n ? What is $\lim_{n\to\infty} a_n$?

E6. Let $a(n,k) = \#\{A \subset [n] : |A| = k, A \text{ does not contain two consecutive elements } \}$.

- **(a)** Prove that

  $$a(n,k) = a(n-1,k) + a(n-2,k-1) \text{ for } k \geq 2$$

  and use it to compute the generating functions $A_k(x) = \sum_{n\geq 1} a(n,k)x^n$.
- **(b)** Use item (a) to prove that

  $$\sum_{k\geq 0} \binom{n-k+1}{k} = F_{n+2}.$$

  Hint: Show that $a(n,k) = \binom{n-k+1}{k}$.

E7. (**Exercise to submit**)

For this exercise, you need to choose between answering parts (a) and (b) or parts (a) and (c). In other words, part (a) is mandatory and then you need to choose between part (b) or part (c).

The Sylvester's money problem consists in studying the amounts that cannot be "paid" using bills of value p and q, where p and q are coprime. Let $A(p,q)$ be the set of linear combinations of p and q with natural coefficients, that is, the amounts that can be paid using p and q:

$$A(p,q) = \{ip + jq : i, j \in \mathbb{N}\}.$$

Since p and q have no common divisors, it can be proven (do not do it) that $A(p,q)$ is partitioned as

$$A(p,q) = \bigcup_{i=0}^{q-1} A_i(p,q),$$

where

$$A_i(p,q) = \{ip + jq : j \in \mathbb{N}\}, \quad \forall 0 \leq i \leq q-1.$$

(a) Let f be the indicator sequence of $A(p,q)$, that is, $f(n) = \mathbf{1}_{A(p,q)}(n)$. Prove that the generating function of f is

$$F(x) = \frac{1 - x^{pq}}{(1-x^p)(1-x^q)}.$$

(b) Let $G(x) = \frac{1}{1-x}$ be the generating function of the indicator function of all natural numbers. The series $H(x) = G(x) - F(x)$ is then equal to the generating function of the indicator function of the values that **cannot** be paid using bills of p and q units. Assuming that $H(x)$ is a polynomial, prove that the largest number outside $A(p,q)$ is $pq - p - q$.

(c) Using the fact that $H(x)$ is a polynomial with coefficients 0 and 1, find the number of natural numbers that do not belong to $A(p,q)$.
Hint: Evaluate H at an appropriate point x.